SCIENTIFIC BOOKS

Leçons sur les Fonctions de Lignes. Professées à la Sorbonne en 1912, par VITO VOLTERRA, recueillies et rédigées par JOSEPH PÉRÈS. Paris, Gauthier Villars, 1913.

The point of view of this book of Volterra's is the systematic generalization of systems of relations of simple type by means of a passage from finite to infinite. We are already familiar with this procedure, in the subject of integral equations, first in the work of Volterra himself, suggested then in the work of Fredholm, and minutely worked out in the papers of Hilbert, his associates and students. But whereas perhaps Hilbert has limited himself to a few aspects of the question and rigorously justified the passage from finite to infinite, considering the subject of forms in an infinite number of variables as a subject for investigation in itself, Volterra has made wide application of an heuristic device for the purpose of obtaining results, which can then sometimes more simply be justified by methods proper to the new subjects themselves. This device is as old as the idea of infinitesimals.

After mentioning the familiar generalizations of this kind, of sum and product, Volterra considers briefly the subject of the generalization of the multiplication of substitutions, coordinate with the integration of linear differential equations, and then devotes the pages of the book proper to the generalization of the idea of function of several variables. This generalization involves the general principles of the study of functional relations.

We are concerned then, in the limit, with the investigation of functions which depend on an infinite number of variables—in particular, on all the points of a curve, or on all the values of another function throughout a certain interval. The general method of making such a study is by procedure from the finite to the infinite.

As an illustration of such a procedure, Volterra cites, in the Introduction, a treatment of the restricted problem of three bodies, by the application of Cauchy's method. The motion of the small body, the only one not known, can be determined by summing the motions obtained by considering the larger bodies as temporarily fixed at various points of their orbits, and proceeding to the limit as these various points on each orbit are taken closer and closer together.

Another passage of the Introduction relates to the definition of the derivative of a function of a curve, and is worth while quoting, since in this case the example is proper to our subject itself. "If a quantity depends upon a curve, we can study the effect produced on the quantity by a variation of the curve. If this variation is very small and limited to the neighborhood of a point of the curve, we arrive at the notion of derivative.[1] For each point of the curve we shall in this way have a derivative. By superposing such variations of the curve, made in all its points, we find the differential, or variation, of the quantity, which will be expressed by means of an integral; in fact, since a function of a curve is a function of an infinite number of variables, the sum which expresses the differential of a function of several variables leads, by the passage to the infinite, to an integral.

"We can then take up the study of differentials of higher order, and thus come to an analytic development analogous to the Taylor's series. The simple double and triple sums, etc., which occur in the development of a function of several variables, are replaced by simple, double, triple, etc., integrals."

The character of this analysis is thus shown. Its purpose is to investigate the phenomena of hysteresis and "evolution" or "heredity" in physical systems—occurrences where the state of the system is supposed to depend upon the history of the system, *i. e.*, to depend upon the values of certain functions throughout all previous instants of time.

In regard to hysteresis and evolution, in physics as in biology, we may adopt two different points of view. One possible standpoint is that the future state of a system is determined entirely by its state at a given instant, and if the history of a system is used in determining its subsequent behavior, that

[1] As the limit, under proper restrictions, of the ratio of the variation of the function to the integral of the variation of the curve, in that neighborhood.

is merely a sign that the instantaneous conditions are insufficiently known. The other point of view is that the history can not be replaced by the consideration of contemporaneous elements; in other words, that a finite number of present elements can not replace the infinity past instants in the determination of the state of the system. The question as to the presence of heredity effects in physical phenomena is, as Volterra points out, of the same character as the old Newtonian question of action at a distance. In fact, if we take account of the theory of relativity and the four-dimensional space-time space, the two questions meet.

Such questions are important if we try to reduce our system of the world to one that is entirely kinetic. In that case we must get rid of the "coefficients of heredity" by explaining them in terms of concealed motions. It may, however, be impossible completely to reduce physical phenomena to a finite number of elements, no matter how described in terms of functions and variables, without exceeding the time limit for speech; and one method may not be more "fundamental" than another. But regardless of our attitude towards the two aspects of the question, or our opinion of the practical value of making such distinctions on the ground of "reality" or "truth," we can not in any case deny the value of the analysis that enables us to take account of such a thing as the history of the system.

Let us turn now to two subjects, elasticity and electricity, where this analysis seems to be usefully introduced. In the usual treatment of elasticity, we have Hooke's law, connecting the deformations and tensions of the system; in electricity the induction and displacement are also connected by linear relations. If now we assume that the tension at any time depends linearly not only on the deformation at that time, but also on the deformation at all previous times, we can introduce this fact into our equations by adding, in our expression of Hooke's law, a term in the form of an integral, whose integrand represents the contribution to the tension at a time t, due to a deformation acting at a time τ through an interval of time $d\tau$. In this way, Hooke's law becomes an integral equation, or a system of integral equations, and the differential equations that determine the deformations or the tensions become integro-differential equations. In a similar way, integro-differential equations are introduced into the subject of electricity.

The study of the methods of integro-differential equations, their solutions, and their applications to the subject of hysteresis, or heredity, form the subject matter of the book from Chapter V. on. In connection with the relative importance which the theory of this subject has assumed in the presentation of Professor Volterra, we may remember that in the case of elasticity it seems to have received important experimental verification in the work of our American physicists, Professor Webster and Dr. Porter.

A detailed analysis of the contents of the book is unnecessary. Some points, however, should be given special mention, because of their universal interest. Chapter IV. is devoted to functional equations in general; that is, to implicit functions of curves. Theorems are obtained which correspond, first to the inversion of an analytic function, and second to the more extensive theorem on the determination of implicit functions in general. In fact it may be noticed that the theorem might be given in such a form as to include the ordinary theorem on implicit functions as a special case, although with respect to the scope of the book such a generalization would be trivial. The condition for the "closed cycle" (Chapter VII.) deserves special attention, because of its relation to the problem of heredity, which, as we have seen, is a central one for the book. In this chapter, section 10 is a first essay at a possible treatment of magnetic hysteresis. Another interesting subject is the application of permutable functions to the solution of integro-differential equations. It is in connection with this subject that are introduced various new sorts of transcendental functions, similar in a way to the exponential function, the sine, and so on. The quality of periodicity, which appears to be lacking,

might be materialized by means of a slightly different sort of symbolism.

The book does not attempt to give a completely exhaustive account of the subject of functions of curves. It omits notable researches by Hadamard, Levy, Fréchet, and confines itself rather closely to the personal researches of the author, who is of course the inventor of their analysis and the principal source of its development. But if it lacks consideration of some of the possible branches, it makes up for the omission by possessing the artistic quality which is characteristic of unified original work. Moreover, the reader will continually find references to theoretical physics and other branches of mathematics, which, besides illuminating profoundly the matter in hand, testify to a not common comprehensiveness of thought on the part of the author.

G. C. Evans

The Essence of Astronomy. By Edward W. Price. G. P. Putnam's Sons. 1914. Pp. xiv + 207. Illustrated.

The Century Dictionary defines *essence* as being the inward nature, true substance, or constitution of anything. From the title of Mr. Price's book, therefore, one would expect to find something of the inward nature of the solar system, or true substance of the stellar universe, some hint as to the underlying causes and formations of the heavens. But one who opens the book with such expectations will be most grievously disappointed, for the work is but a compilation of the simplest statistical facts; facts which have been compiled and written about over and over again. Further, the book contains some strange and new conceptions: to classify the milky way as a freak, and double and variable stars as oddities, is certainly new, and such classification, itself, might even be called odd and freakish.

The book is well made mechanically, well printed, with clear and beautiful illustrations, but otherwise it is one of dozens of similar crude compilations.

Chas. Lane Poor

An Introduction to General Psychology. By Robert Morris Ogden. Longmans, Green and Co., 1914. Pp. xviii + 270.

Professor Ogden's text-book is the outcome of a definite abandonment of the purely sensationalistic conception of psychology. Dr. Ogden defines his science as "the study of mental happenings." He treats not merely of "mental contents" and their physical conditions, but also of the "mental activities" which constitute what he rather vaguely calls the "purposive aspect" of mental happenings. As elements of mental contents Dr. Ogden enumerates sensations, images, thoughts—which he classifies as notions or relations—and affections. Attention, memory, perception, ideation, emotion and reaction are brought together under the heading "The Synthetic Facts of Mind." The concluding section of the book contains chapters on "mind and body," "personality" and "character." In the last of these chapters Mr. Ogden suggests the relation of psychology to logic, to esthetics, to ethics and to religion. Under the second heading he discusses mainly sleep, dreams, hypnosis, multiple personality and insanity. Not all teachers—it may be noted—will approve the inclusion of the topics just named in a book of fewer than 300 pages; and many will regret the brevity with which all topics are treated and the omission of "all diagrams, references to literature and practical demonstrations."

The writer of this notice is glad to find Professor Ogden in substantial agreement with Herbert Spencer, William James, Binet, Meinong, the Würzburg school, and with several recent American writers in his view that thought-elements as well as sensational and affective elements, should be explicitly acknowledged in a text-book of psychology; and she welcomes also his repeated descriptions of consciousness—the relating consciousness (pp. 14 ff.), affection (pp. 85 ff.) and will (pp. 171 f.)—in terms of the self who is conscious. Occasional artificial constructions and a certain vagueness in the use of the term "mental activity" might indeed have been avoided, had this natural and inevitable point of view been more steadily held.